

Effects of noisy entangled state transmission on quantum teleportation

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Abstract

Quantum teleportation requires the transmission of entangled pairs to Alice and Bob. Transmission errors modify the entangled state before the teleportation can be performed. We determine the changes in the output state caused by such transmission errors. It is shown that the errors caused by entanglement transmission are equivalent to the errors caused in a direct transmission of a quantum state from Alice to Bob.

Keywords:

quantum communication, teleportation fidelity

1 Introduction

Quantum teleportation is an entanglement assisted method for transferring quantum information [1]. Since teleportation strategies have been realized experimentally [2, 3], it is now of interest to discuss possible applications and their technical limitations. If quantum teleportation is actually applied to transfer quantum information over large distances, one of the major sources of errors will be the quantum communication lines used for entanglement distribution. In the following, we present a general formalism by which the effect of a specific type of error in the entanglement distribution on the output state of quantum teleportation can be predicted. In section 2, we present a formulation of ideal quantum teleportation for any N-level system. In section 3, the general error transfer properties are analyzed. Finally, the result is analyzed in section 4 and conclusions are drawn in section 5.

2 Ideal quantum teleportation

Quantum teleportation can be applied to any N-level quantum system. Ideally, it requires a maximally en-

tangled state. Given an appropriate basis $|n\rangle$, such a quantum state can always be written as

$$|E_{\max}\rangle_{i,j} = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} |n\rangle_i \otimes |n\rangle_j. \quad (1)$$

The second requirement for quantum teleportation is the ability to carry out a measurement projecting a pair of systems onto a maximally entangled state. Ideally, each measurement result m corresponds to a maximally entangled state,

$$|P(m)\rangle_{i,j} = \sqrt{\frac{\chi(m)}{N}} \sum_{n=0}^{N-1} \left(\hat{U}(m) |n\rangle \right)_i \otimes |n\rangle_j, \quad (2)$$

where the normalization factor $\chi(m)$ and the unitary transformation $\hat{U}(m)$ fulfill the conditions for a positive operator valued measure,

$$\sum_m |P(m)\rangle\langle P(m)| = \hat{1}. \quad (3)$$

Note that this requirement is only necessary for unconditional teleportation. In conditional teleportation, only some measurement results m correspond to maximally entangled states. The following analysis applies to both conditional and unconditional teleportation. However, the loss of quantum information due to a conditional teleportation may easily be a more serious source of error than the transmission errors considered here.

For a specific measurement result m , the principle of quantum teleportation may now be represented schematically by assigning the input state to system A and the entanglement to a reference system R and the remote system B . The measurement is performed on the joint input from system A and the reference system R . As a result of the entanglement, B is then projected

into a unitary transformation of the input state $|\psi_{\text{in}}\rangle$ conditioned by the measurement result m , as given by

Initial state

$$\frac{1}{\sqrt{N}} \sum_n |\psi_{\text{in}}\rangle_A \otimes |n\rangle_R \otimes |n\rangle_B$$

Measurement projection

$$\sqrt{\frac{\chi(m)}{N}} \sum_n \left(\langle n | \hat{U}^{-1}(m) \right)_A \otimes \langle n |_R$$

Conditional output state

$$\begin{aligned} & \frac{\sqrt{\chi(m)}}{N} \sum_n \langle n | \hat{U}^{-1}(m) | \psi_{\text{in}} \rangle \quad |n\rangle_B. \\ & = \frac{\sqrt{\chi(m)}}{N} \hat{U}^{-1}(m) | \psi_{\text{in}} \rangle_B \end{aligned} \quad (4)$$

It should be noted that, at this point, no information has been transferred from A to B and the density matrix of an observer in B is still in a maximally mixed state as long as the measurement result m is not known. Teleportation is completed by classically communicating the measurement result m to B . The unitary transformation $\hat{U}^{-1}(m)$ can then be compensated by performing $\hat{U}(m)$ on the output to restore the original state $|\psi_{\text{in}}\rangle$ in B .

3 Error transfer

For long distance teleportation, the most important technical requirement is the distribution of entanglement over long distances. Transmission errors act randomly on the transmission lines for systems R and B . However, all such random errors can be represented as mixtures of quantum mechanically precise operators $\hat{F}_R(x_r)$ and $\hat{F}_B(x_b)$, where x_r and x_b are the random variables defining the type of error. The probability distribution over x_r and x_b is given by the expectation values of $\hat{F}_i^\dagger(x_i)\hat{F}_i(x_i)$, requiring that

$$\sum_{x_i} \hat{F}_i^\dagger(x_i)\hat{F}_i(x_i) = \hat{1}. \quad (5)$$

Figure 1 illustrates the effect of a well defined pair of errors $\hat{F}_R(x_r)$ and $\hat{F}_B(x_b)$ on the teleportation process. The total error can be represented by an output error operator $\hat{F}_{\text{out}}(m; x_r, x_b)$. This operator can be derived from the teleportation scheme introduced in section 2.

With the errors, the initial state for teleportation reads

$$\frac{1}{\sqrt{N}} \sum_n |\psi_{\text{in}}\rangle_A \otimes \hat{F}_R(x_r) |n\rangle_R \otimes \hat{F}_B(x_b) |n\rangle_B. \quad (6)$$

By replacing the ideal initial state in equation (4), the outcome of the teleportation measurement of m can be

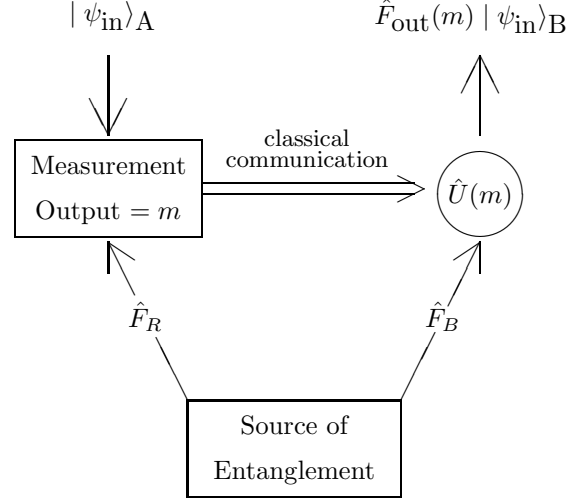


Figure 1: Schematic representations of entanglement distribution errors in quantum teleportation.

calculated. It reads

$$\begin{aligned} & \frac{\sqrt{\chi(m)}}{N} \sum_{n,n'} \langle n' | \hat{U}^{-1}(m) | \psi_{\text{in}} \rangle \\ & \times \langle n' | \hat{F}_R(x_r) | n \rangle \hat{F}_B(x_b) | n \rangle_B. \end{aligned} \quad (7)$$

This equation may be greatly simplified by introducing the transpose $\hat{F}_R^T(x_r)$ of $\hat{F}_R(x_r)$ in the $|n\rangle$ basis, defined by

$$\langle n' | \hat{F}_R(x_r) | n \rangle = \langle n | \hat{F}_R^T(x_r) | n' \rangle. \quad (8)$$

With this definition, the output state of the teleportation after the application of $\hat{U}(m)$ reads

$$\frac{\sqrt{\chi(m)}}{N} \hat{U}(m) \hat{F}_B(x_b) \hat{F}_R^T(x_r) \hat{U}^{-1}(m) | \psi_{\text{in}} \rangle_B. \quad (9)$$

The difference between this teleportation result and the ideal result is described by the effective error operator $\hat{F}_{\text{out}}(m; x_r, x_b)$, given by

$$\hat{F}_{\text{out}}(m; x_r, x_b) = \hat{U}(m) \hat{F}_B(x_b) \hat{F}_R^T(x_r) \hat{U}^{-1}(m). \quad (10)$$

With this equation, it is possible to characterize the teleportation errors based on any combination of transmission errors $\hat{F}_R(x_r)$ and $\hat{F}_B(x_b)$. In general, the errors are only changed by unitary transformations, indicating that any properties invariant under such transformations will be preserved. All other properties of an error are transformed according to the measurement result m obtained in the teleportation process. Knowledge of m is therefore important in order to minimize the effects of errors in the teleportation process [4].

4 Properties of the output error

The result shows that the errors in R and the errors in B are effectively multiplied, representing the expected exponential increase in errors with the length of the communication line. Indeed, the total error may be interpreted as a sequential error of an initial transmission through channel R with an error of $\hat{U}(m)\hat{F}_R^T(x_r)\hat{U}^{-1}(m)$, followed by a transmission through channel B with an error of $\hat{U}(m)\hat{F}_B(x_b)\hat{U}^{-1}(m)$. The only differences between the teleportation and the direct transmission of the quantum state through the transmission lines used for R and B are given by the unitary transformation $\hat{U}(m)$ and the transposition operation on the error in R . However, this transposition is also a unitary transformation that preserves the general properties of $\hat{F}_R(x_r)$, such as the eigenvalues or the overlap between eigenstates. From this property, it is clear that the channel capacity of quantum teleportation is exactly identical with the capacity for a direct transmission. It is for this reason, that the use of teleportation as a quantum repeater requires some form of reliable entanglement purification [5]. Nevertheless it is interesting to note that the errors are not increased by the use of entanglement in the channels. In this context, entanglement is no more sensitive to errors than any arbitrary quantum state.

It is now possible to consider different types of error statistics. The most simple errors are homogeneous ones, such that any unitary transformation $\hat{U}(m)$ merely causes a permutation of the errors x_i ,

$$\hat{U}(m)\hat{F}_i(x_i)\hat{U}^{-1}(m) = \hat{F}_i(x'_i). \quad (11)$$

If the x_i are unknown, such errors are independent of m and the teleportation errors correspond to the direct transmission except for the transposition in R . However, it is likely that errors homogeneous with respect to $\hat{U}(m)$ are also homogeneous with respect to the transposition,

$$\hat{F}_R^T(x_r) = \hat{F}_R(x'_r). \quad (12)$$

In fact, a much wider class of errors should fulfill this requirement, making it possible to identify the transposed errors with non-transposed ones.

In order to illustrate the physics of a transposition, it is helpful to consider the angular momentum operators \hat{l}_x, \hat{l}_y , and \hat{l}_z . A transposition in the basis of \hat{l}_z eigenstates leaves both \hat{l}_x and \hat{l}_z invariant, while reversing the sign of $\hat{l}_y^T = -\hat{l}_y$. The transposition therefore corresponds to a mirror image in the xz -plane. For reasons of symmetry it is likely that transmission errors will indeed be homogeneous with respect to this operation. In the case of transmission lines with a non-homogeneous error, the transposition will effectively reverse the asymmetry.

For errors which are not homogeneous with respect to $\hat{U}(m)$, the output error statistics depend strongly on the measurement result m . Equation (10) then describes the correlation between m and the output error. If the output states are analyzed without reference to m , the teleportation will be far more noisy than a direct transmission. However, this type of noise originates from a classical randomization of the signal based on the known variable m . This noise source can therefore be eliminated by analyzing the full correlated statistics.

5 Conclusions

The entanglement distribution errors are essentially equivalent to errors in a direct transmission of quantum states through the same communication lines. If the errors are homogeneously distributed, it makes no difference whether the quantum state is teleported or it is sent directly. For non-homogeneous errors, the teleportation measurement result m represents an additional source of randomness. However, the random variable m is known to both the sender and the receiver, indicating that this randomness will not cause any loss of information. In conclusion, the errors caused by entanglement distribution are remarkably similar to the errors of a direct transmission through the same quantum channels. The method of teleportation itself neither reduces nor enhances the errors.

References

- [1] C.H. Bennet, G. Brassard, C. Crepeau, R.Jozsa, A. Peres, and W.K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
- [2] D. Bouwmeester, J.-W. Pan, K.Mattle, M.Eibl, H. Weinfurter, and A. Zeilinger, Nature (London) **390**, 575 (1997).
- [3] A. Furusawa, J.L. Sorensen, S.L. Braunstein, C.A. Fuchs, H.J. Kimble, and E.S. Polzik, Science **282**, 706 (1998).
- [4] H.F. Hofmann, T. Ide, T. Kobayashi, and A. Furusawa, Phys. Rev. A **62**, 062304 (2000).
- [5] W.Dür, H.-J. Briegel, J.I. Cirac, and P. Zoller, Phys. Rev. A **59**, 169 (1999).